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ON A METHOD OF STUDYING NONLINEAR OSCILLATIONS OF VISCOELASTIC SYSTEMS

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We propose a method of constructing approximately the solutions of equations which describe nonlinear oscillations of viscoelastic systems. In contrast to what was done in [1−3], we present the method of a direct constructing the solutions of integro-partial differential equations without first having to reduce them to ordinary equations with a subsequent reduction to standard form. As is shown in (3−5], oscillations of viscoelastic systems are described by nonlinear integro-partial differential equations. The method proposed in [1−3] for studying such equations by applying the Bubnov-Galerkin method or the method of lines involved in the reduction of these equations to ordinary differential equations, setting them into a standard form, and then carrying out an averaging process.

We present below a direct method for the construction of asymptotic expansions of the solutions of the corresponding equations of viscoelasticity (*).

1. We begin with the consideration of linear problems of the dynamic theory of viscoelasticity. The dynamic equations of the linear theory of viscoelasticity have the form [5, 6]

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho F + \mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} - \tag{1.1}$$

$$\varepsilon \mu^* \Delta \mathbf{u} - \varepsilon (\lambda^* + \mu^*) \operatorname{grad} \operatorname{div} \mathbf{u}, \quad \mathbf{u} = (u_1, u_2, u_3)$$

$$\mu^* \varphi(t, \mathrm{x}) = \frac{1}{2} \int_0^t \Gamma(t - \tau) \varphi(\tau, \mathrm{x})\, d\tau$$

$$\lambda^* \varphi(t, \mathrm{x}) = \int_0^t \Gamma_0(t - \tau) \varphi(\tau, \mathrm{x})\, d\tau$$

These equations must be supplemented with appropriate initial and boundary conditions.

In [3, 4] it is shown that for load-carrying structures of polymer materials the parameter ε can be regarded as small. By analogy with [7, 8] we seek a solution of the system (1.1) in the following form:

$$\mathbf{u}(t, x) = a(t) \varphi(\mathrm{x}) \cos \theta(t) + \varepsilon u_1(a, \theta, \mathrm{x}) + O(\varepsilon^2) \tag{1.2}$$

Here $\varphi(\mathrm{x})$ is a vector-valued function of the vector argument $\mathrm{x} = (x_1, x_2, x_3)$, and the functions $a(t)$ and $\theta(t)$ are determined from the equations

$$a^{\cdot} = \varepsilon A_1(a) + O(\varepsilon^2), \quad \theta^{\cdot} = \omega + \varepsilon B_1(a) + O(\varepsilon^2) \tag{1.3}$$

The problem amounts to finding a way to determine the functions A_1, B_1 and u_1. Before proceeding in this direction, we carry out the following calculations. In accord with

*) The problem of constructing directly the solutions of the equations of viscoelasticity was posed by A. A. Il'iushin.

the relations (1.2) and (1.3) we obtain

$$\partial^2\mathbf{u}/\partial t^2 = -a\varphi\omega^2\cos\theta + \varepsilon\left(\omega^2\frac{\partial^2 u_1}{\partial\theta^2} - 2A_1\varphi\omega\sin\theta - 2B_1 a\varphi'\omega\cos\theta\right) + O(\varepsilon) \qquad (1.4)$$

Further, we can show, under fairly general assumptions, that

$$a(t-\tau) = a(t) + O(\varepsilon\tau), \qquad \theta(t-\tau) = \theta(t) - \omega(\tau) + O(\varepsilon\tau)$$

Therefore assuming that

$$\int_0^\infty \Gamma(s)\, s ds < \text{const}, \qquad \int_0^\infty \Gamma_0(s)\, s ds < \text{const}$$

$$\int_{\alpha/\varepsilon}^\infty \Gamma(s)\, ds = O(\varepsilon), \qquad \int_{\alpha/\varepsilon}^\infty \Gamma_0(s)\, ds = O(\varepsilon), \qquad \varepsilon\to 0, \qquad \alpha > 0$$

we find

$$\varepsilon\mu^*\Delta\mathbf{u} = \frac{\varepsilon}{2}a\Delta\varphi(M\cos\theta + N\sin\theta) + O(\varepsilon) \qquad (1.5)$$

$$\varepsilon(\lambda^* + \mu^*)\,\mathrm{grad\,div}\,\mathbf{u} = \varepsilon a\,\mathrm{grad\,div}\,\varphi\left[\left(P + \frac{M}{2}\right)\cos\theta + \left(Q + \frac{N}{2}\right)\sin\theta\right] + O(\varepsilon)$$

$$M = \int_0^\infty \Gamma(s)\cos\omega s ds, \qquad N = \int_0^\infty \Gamma(s)\sin\omega s ds$$

$$P = \int_0^\infty \Gamma_0(s)\cos\omega s ds, \qquad Q = \int_0^\infty \Gamma_0(s)\sin\omega s ds$$

Substituting the relation (1.2) into Eq. (1.1) and taking into account the relations (1.4) and (1.5), we obtain, to within quantities of order ε ,

$$\begin{aligned} -\rho a\varphi\omega^2\cos\theta + \varepsilon\left(\rho\omega^2\frac{\partial^2 u_1}{\partial\theta^2} - 2\rho A_1\varphi\omega\sin\theta - 2\rho B_1 a\varphi\omega\cos\theta\right) = \\ [\mu\Delta\varphi + (\lambda+\mu)\,\mathrm{grad\,div}\,\varphi]\, a\cos\theta + \varepsilon[\mu\Delta u_1 + (\lambda+\mu)\,\mathrm{grad\,div}\,u_1] - \\ \frac{\varepsilon}{2}a\Delta\varphi(M\cos\theta + N\sin\theta) - \varepsilon a\,\mathrm{grad\,div}\left[\left(P+\frac{M}{2}\right)\cos\theta + \left(Q + \frac{N}{2}\right)\sin\theta\right] \end{aligned} \qquad (1.6)$$

The determination of the functions A_1, B_1 and u_1 from Eq. (1.6) in the general case requires a complicated procedure. Therefore we describe here only one of the possible ways of solving this problem. For example, if we determine the function $\varphi(x)$ as the solution of the equation

$$\mu\Delta\varphi + (\lambda+\mu)\,\mathrm{grad\,div}\,\varphi + \rho a\omega^2\varphi = 0 \qquad (1.7)$$

then for determining the functions A_1 and B_1 we can proceed as follows. We multiply Eq. (1.6) scalarly by $\varphi(\mathbf{x})\sin\theta$ and integrate with respect to x over the surface s and with respect to θ from 0 to 2π. Next we multiply (1.6) by $\varphi(\mathbf{x})\cos\theta$ and follow the same procedure. For the determination of u_1 we can also use the equation

$$\rho\omega^2\frac{\partial^2 u_1}{\partial\theta^2} = \mu\Delta u_1 + (\lambda+\mu)\,\mathrm{grad\,div}\,u_1$$

We give an example. Consider the longitudinal oscillations of a viscoelastic rod of length l

$$\frac{\partial^2\mathbf{u}}{\partial\mathbf{x}^2} - p^2\frac{\partial^2\mathbf{u}}{\partial t^2} = \varepsilon\int_0^t \Gamma(t-\tau)\frac{\partial^2\mathbf{u}}{\partial\mathbf{x}^2}d\tau \qquad (1.8)$$

$$\mathbf{u}|_{x=0} = 0, \qquad \left.\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right|_{x=0} = 0, \qquad \mathbf{u}|_{t=0} = f_1(\mathbf{x},\varepsilon), \qquad \left.\frac{\partial\mathbf{u}}{\partial t}\right|_{t=0} = f_2(\mathbf{x},\varepsilon)$$

In this case Eqs. (1.6) and (1.7) assume the form

$$\varphi''(x) + p^2\omega^2\varphi = 0 \tag{1.9}$$
$$\frac{\partial^2 u}{\partial x^2} - p^2\omega^2 \frac{\partial^2 u_1}{\partial \theta^2} + 2p^2\omega A_1\varphi \sin\theta + 2p^2\omega a B_1 \varphi \cos\theta = a\varphi''(M\cos\theta + N\sin\theta)$$

From the first of Eqs. (1.9) we obtain

$$\varphi(x) = \sum_{n=1}^{\infty} \alpha_n \sin(p\omega_n x), \qquad \varphi(0) = \varphi'(l) = 0$$
$$p\omega_n = \frac{(2n-1)}{2l}\pi, \qquad n = 1, 2, \ldots$$

We denote the functions A_1, B_1, a and θ by a subscript n. Multiplying the second of Eqs. (1.9) first by $\varphi \cos\theta_n$, and then by $\varphi \sin\theta_n$ and integrating the result with respect to x and θ_n over the intervals $0 \leqslant x \leqslant l$, $0 \leqslant \theta_n \leqslant 2\pi$, we find

$$A_{1n} = {}^1/_2 N\omega_n a_n, \qquad B_{1n} = {}^1/_2 M_n\omega_n$$
$$M_n = \int_0^\infty \Gamma(s)\cos\omega_n s ds, \qquad N_n = \int_0^\infty \Gamma(s)\sin\omega_n s ds$$

Integrating now Eqs. (1.3), we obtain a_n and θ_n

$$a_n = a_{0n}\exp\left(-\frac{\varepsilon}{2}N_n\omega_n t\right), \qquad \theta_n = \omega_n\left(1 - \frac{\varepsilon}{2}M_n\right)t + \theta_{0n}$$

Finally, we have

$$u(x, t) = \sum_{n=1}^{\infty} \beta_n \exp\left(-\frac{\varepsilon}{2}N_n\omega_n t\right)\cos\left[\omega_n\left(1 - \frac{\varepsilon}{2}M_n\right)t + \theta_{0n}\right]\sin\omega_n x$$

(the parameters β_n and θ_{0n} are obtained from the initial conditions).

2. We proceed now to consider the equations of nonlinear viscoelasticity obtained in [3]. Assuming that the functions R_{*1} and R_{*2} in these equations are independent of z and that the integral terms are proportional to a small parameter, we can extend the method of constructing solutions, as described anove, to equations of this kind.

Let us illustrate this method applying it to problem (1.8) in which the right-hand side is supplemented by the nonlinear term

$$I = \varepsilon \int_0^t G(t-\tau)\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} d\tau$$

Since the calculations here are analogous to the previous ones, we give only the transformations of the nonlinear term I. Using the expansion (1.3) and (1.4), we find, as above, to within terms of order ε

$$I = \varepsilon \int_0^{t'} G(t-\tau)\, a^3\varphi'^2\varphi'' \cos^3\theta d\tau + O(\varepsilon) \approx \frac{3}{4}\varepsilon a^3\varphi'^2\varphi''(C\cos\theta + D\sin\theta) + O(\varepsilon)$$
$$C = \int_0^\infty G(s)\cos\omega s ds, \qquad D = \int_0^\infty G(s)\sin\omega s ds$$

We note that in calculating the integral term only, terms containing the first harmonics are retained here. In the general case, in the expansion of the integrand function in a Fourier series we can include the terms containing the higher harmonics in the equation from which the function u_1 is determined. Carrying out the procedure described above, we obtain equations for the determination of a and θ, which are easily integrated. The subsequent calculations are obvious.

REFERENCES

1. Filatov, A. N., Methods of Averaging in Differential and Integro-Differential Equations. "Fan", Tashkent, 1971.
2. Mitropol'skii, Iu. A. and Filatov, A. N., Averaging of integro-differential and integral equations. Ukrainsk. Matem. Zh., Vol. 24, №1, 1972.
3. Il'iushin, A. A., Movliankulov, Kh., Sunchaliev, R. M., and Filatov, A. N., On some methods for studying nonlinear problems of viscoelasticity theory. Dokl. Akad. Nauk SSSR, Vol. 206, №1, 1972.
4. Il'iushin, A. A. and Ogibalov, P. M., Quasilinear theory of voscoelasticity and the small parameter method. Mekhanika Polimerov, №2, 1966.
5. Il'iushin, A. A. and Pobedria, B. K., Fundamentals of the Mathematical Theory of Thermoviscoelasticity. "Nauka", Moscow, 1970.
6. Il'iushin, A. A., Mechanics of a Continuous Medium. Moscow State University, Moscow, 1971.
7. Mitropol'skii, Iu. A., Method of Averaging in Nonlinear Mechanics. "Naukova Dumka", Kiev, 1971.
8. Mitropol'skii, Iu. A. and Moseennov, B. I., Lectures on the Application of Asymptotic Methods to the Solution of Partial Differential Equations. Academy of Sciences of the Ukrainian SSR, Kiev, 1963.

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PURE SHEAR OF AN ELASTIC HALFSPACE WITH A SYSTEM OF CRACKS

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We consider a dynamic mixed problem for an elastic halfspace, weakened by a system of two-dimensional cracks and subject to conditions of anti-plane deformation.

We raise the problem of determining the jump in the stresses at the cracks in an elastic halfspace when shear displacements on the cracks are known. Using the method developed in [1, 2] we reduce the system of integral equations for the mixed problem to an equivalent system of linear algebraic equations with a completely continuous operator. We analyze the problems relating to the solvability of the integral equations and the infinite system. Investigation of the solution in the zero approximation is given.

The dynamics of an elastic halfspace with a crack was studied in [3, 4] wherein the main emphasis was focused on problems relating to crack propagation and the diffraction of elastic waves by the cracks.